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TRANSVERSAL PACKINGS AND COVERS

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PREFACE

As part of its Project RAND research program, RAND engages in basic supporting studies in mathematics. The present Memorandum treats two combinatorial problems involving partial transversals of a collection of subsets of a finite set.

SUMMARY

Let $S_1, S_2, \cdots, S_m$ be a collection of subsets of a finite set S. The subset T of S is a partial transversal of size t of the subsets $S_1, S_2, \cdots, S_m$ if (i) T consists of t elements of S, and (ii) the term rank of the incidence matrix of $S_1, S_2, \cdots, S_m$ vs. elements of T is equal to t. Necessary and sufficient conditions in order that $S_1, S_2, \cdots, S_m$ have mutually disjoint partial transversals $T_1, T_2, \cdots, T_p$ of sizes $t_1, t_2, \cdots, t_p$ are known. This Memorandum shows that the problem of constructing such a transversal packing can be formulated in terms of flows in networks, and existence conditions obtained from results in network flow theory. A similar approach is presented for an analogous covering problem involving partial transversals of prescribed sizes. Finally, connections are indicated between these problems and some recent results of Edmonds in matroid theory.

TRANSVERSAL PACKINGS AND COVERS

1. INTRODUCTION

Let $S_1, S_2, \cdots, S_m$ be a finite collection of (not necessarily distinct) subsets of a finite set $S = \{e_1, e_2, \cdots, e_n\}$. The set $T = \{e_{j_1}, e_{j_2}, \cdots, e_{j_t}\}$, $0 < t \leq n$, is a partial transversal (of size t) of the subsets $S_1, S_2, \cdots, S_m$ if (i) $e_{j_1}, e_{j_2}, \cdots, e_{j_t}$ are distinct elements of S, and (ii) there are distinct integers $i_1, i_2, \cdots, i_t$ such that $e_{j_k} \in S_{i_k}$, $k = 1, 2, \cdots, t$. Higgins [3] has found necessary and sufficient conditions in order that the subsets $S_1, S_2, \cdots, S_m$ have mutually disjoint partial transversals $T_1, T_2, \cdots, T_p$ of respective sizes $t_1, t_2, \cdots, t_p$. It is shown in [3] that such transversals exist if and only if, for $k = 0, 1, \cdots, m$, every k of the subsets $S_1, S_2, \cdots, S_m$ contain between them at least

$$(1.1) \qquad \alpha_k = \sum_{j=m-k+1}^{\infty} t_j^*$$

distinct elements. In (1.1), the integer t_j^* is the number of integers t_i satisfying $t_i \geq j$, i.e., $[t_i]$ and $[t_j^*]$ are conjugate partitions of $\sum_{i=1}^{p} t_i$.

The proof given in [3] of this theorem uses an intricate induction. In Sec. 2 we describe another approach, via

flows in networks [2], which yields Higgins' result on transversal packing and provides a good algorithm for constructing the packing or showing none exists. Another set of necessary and sufficient conditions is also deduced. Section 3 presents a similar approach to an analogous covering problem involving partial transversals of prescribed sizes. It is shown that the subsets $S_1, S_2, \cdots, S_m$ have partial transversals $T_1, T_2, \cdots, T_p$ of sizes $t_1, t_2, \cdots, t_p$ whose union $\bigcup_{i=1}^{p} T_i$ includes all elements of S if and only if (i) no t_i exceeds the maximum possible transversal size ρ, and (ii) for every $S' \subseteq S$,

$$(1.2) \qquad |S'| \leq \sum_{j=1}^{r(S')} t_j^*.$$

In (1.2), $|S'|$ denotes the number of elements in S' and $r(S')$ is the number of subsets among $S_1, S_2, \cdots, S_m$ which contain some element of S'.

Special cases of the foregoing packing and covering problems arise by taking $t_i = t$, $i = 1, 2, \cdots, p$. One can then ask for a maximum packing and a minimum cover composed of partial transversals of size t. The resulting theorems are closely related to two recent theorems of Edmonds [1] concerning maximum packings and minimum covers composed of bases in matroids. This is discussed in Secs. 4 and 5.

2. TRANSVERSAL PACKING

The packing problem for partial transversals of prescribed sizes can be posed in terms of maximal flow from source to sink in a suitable directed network having capacity constraints on arcs. We refer to [2] for the notation, definitions, and theorems about network flows which will be used in this and the following section.

A representing network for transversal packing is shown in Fig. 2.1. In Fig. 2.1 we have, in addition to

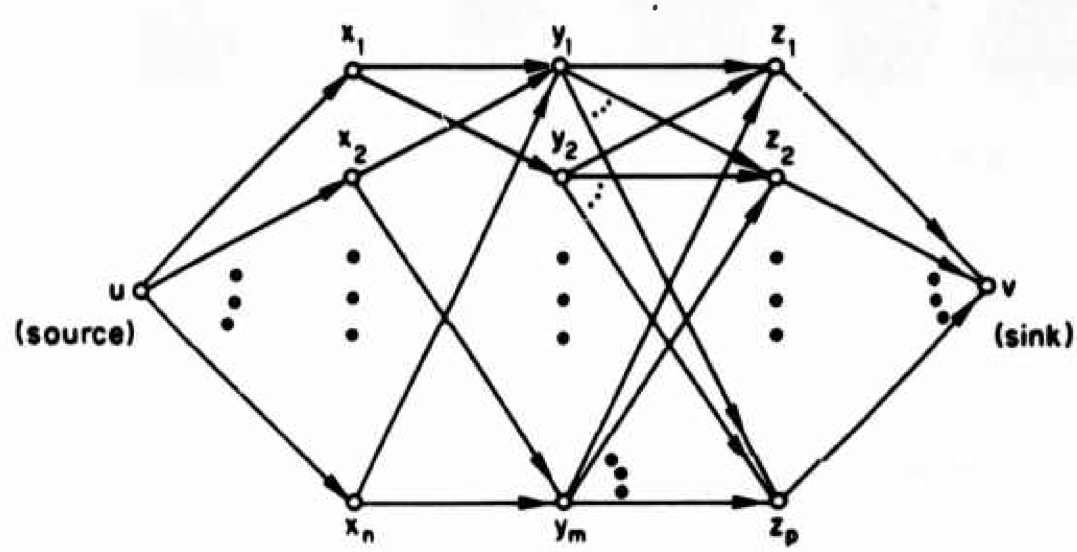

Fig. 2.1

a source u and sink v, three tiers of nodes: $x_1, x_2, \cdots, x_n$ (corresponding to elements $e_1, e_2, \cdots, e_n$ of the set S); $y_1, y_2, \cdots, y_m$ (corresponding to the subsets $S_1, S_2, \cdots, S_m$ of S); and $z_1, z_2, \cdots, z_p$ (corresponding to the transversals $T_1, T_2, \cdots, T_p$). The directed arcs of Fig. 2.1 and their flow capacities are listed below:

Arcs	Capacities
(u,x_j), $j = 1,2,\cdots,n$	$c(u,x_j) = 1$
(x_j,y_i) corresponding to $e_j \epsilon S_i$	$c(x_j,y_i) = \infty$
(y_i,z_k), $i = 1,2,\cdots,m$; $k=1,2,\cdots,p$	$c(y_i,z_k) = 1$
(z_k,v), $k = 1,2,\cdots,p$	$c(z_k,v) = t_k$

An integral flow f from source to sink in this network which satisfies

$$(2.1) \qquad f(z_k,v) = c(z_k,v) = t_k, \; k = 1,2,\cdots,p,$$

produces mutually disjoint transversals of sizes $t_1, t_2, \cdots, t_p$ in the following manner. Take a chain decomposition [2; Sec. I-2] of the flow f and put e_j in T_k if, for some $i = 1, 2, \cdots, m$, the arcs (x_j, y_i), (y_i, z_k) occur in some chain of this decomposition. Conversely, mutually disjoint partial transversals of sizes $t_1, t_2, \cdots, t_p$ produce an integral flow from source to sink satisfying (2.1). Using the

integrity theorem for network flows [2; Th. 8.1], it follows that the desired packing exists if and only if the value of a maximal flow from u to v in the representing network of Fig. 2.1 is equal to $t_1 + t_2 + \cdots + t_p$. By the max-flow min-cut theorem [2; Th. 5.1] this will be so if and only if the capacity of each cut separating u and v is at least $t_1 + t_2 + \cdots + t_p$.

To analyze this condition in more detail, let X, Y, Z be arbitrary subsets of $\{x_1, x_2, \cdots, x_n\}$, $\{y_1, y_2, \cdots, y_m\}$, and $\{z_1, z_2, \cdots, z_p\}$, respectively, and denote their complements in these sets by $\bar{X}, \bar{Y}, \bar{Z}$. Then the transversal packing exists if and only if the inequalities

$$(2.2) \quad \sum_{x_j \in \bar{X}} c(u,x_j) + \sum_{\substack{x_j \in X \\ y_i \in \bar{Y}}} c(x_j,y_i) + \sum_{\substack{y_i \in Y \\ z_k \in \bar{Z}}} c(y_i,z_k) + \sum_{z_k \in Z} c(z_k,v) \geq \sum_{k=1}^{p} t_k$$

hold for all such X, Y, Z. Using the table of arc capacities, (2.2) reduces to

$$(2.3) \qquad |\bar{X}| + |Y| \cdot |\bar{Z}| \geq \sum_{z_k \in \bar{Z}} t_k$$

for all X, Y, Z such that the set of arcs $(X,\bar{Y})$ leading from nodes of X to nodes of $\bar{Y}$ is empty. (Otherwise the second sum in (2.2) is infinite and (2.2) holds

automatically.) Let $B(\bar{Y})$ denote the nodes of $\{x_1, x_2, \cdots, x_n\}$ which have arcs leading to nodes of $\bar{Y}$. Since $(X, \bar{Y})$ is empty, we have $B(\bar{Y}) \subseteq \bar{X}$ and hence we may take $\bar{X} = B(\bar{Y})$. Moreover, as $\bar{Z}$ ranges over those subsets of $\{z_1, z_2, \cdots, z_p\}$ which have fixed cardinality $|\bar{Z}| = e$, the left hand side of (2.3) is fixed and hence we may take $\bar{Z}$ to correspond to the e largest t_k. Thus, selecting the notation so that

$$(2.4) \qquad t_1 \geq t_2 \geq \cdots \geq t_p > 0,$$

we have $\bar{Z} = \{z_1, z_2, \cdots, z_e\}$. With these simplifications, the inequalities (2.3) become, in set-theoretic terms,

$$(2.5) \qquad \left|\bigcup_{i\in I} S_i\right| \geq \sum_{j=1}^{e} t_j - e(m - |I|)$$

for $e = 0, 1, \cdots, p$ and all $I \subseteq \{1, 2, \cdots, m\}$. Thus the desired transversal packing exists if and only if the inequalities (2.5) hold.

To deduce Higgins' condition from (2.5), take $|I| = k$ and observe that

$$(2.6) \qquad \max_{0\leq e\leq p} \left[\sum_{j=1}^{e} t_j - e(m - k)\right] = \sum_{j=m-k+1}^{\infty} t_j^*.$$

This is perhaps most readily seen in terms of a partition diagram.

Theorem 2.1 (Higgins). The subsets $S_1, S_2, \cdots, S_m$ of S have mutually disjoint partial transversals of sizes $t_1, t_2, \cdots, t_p$ if and only if the union of every k of the subsets, for $k = 0, 1, \cdots, m$, contains at least $\alpha_k = \sum_{j=m-k+1}^{\infty} t_j^*$ distinct elements.

Of more interest perhaps is the fact that the above proof indicates that a good algorithm is available for constructing a packing of transversals, since the "labeling method" [2; Sec. I-8] for constructing maximal flows in networks applies.

If we return to (2.3) and take a slightly different route, eliminating Y rather than X, another set of necessary and sufficient conditions results. In (2.3) we have $(X,\bar{Y})$ empty. Letting $A(X)$ denote the set of nodes which are terminals of arcs originating in X, we thus have $A(X) \subseteq Y$, and hence we may take $Y = A(X)$. Following the same reasoning as before, we are led to the conditions

$$(2.7)\quad |\bar{X}| \geq \sum_{i=1}^{e} t_i - e\cdot|A(X)|,\ e = 0,1,\cdots p,\ X \subseteq \{x_1 x_2 \cdots, x_n\},$$

as necessary and sufficient for the existence of the required flow.

If we let $r(S')$ denote the number of subsets among $S_1, S_2, \cdots, S_m$ which contain some element of $S' \subseteq S$ (that is, the number of subsets "represented" by S'), we

then have:

<u>Theorem</u> 2.2. <u>The subsets</u> $S_1, S_2, \cdots, S_m$ <u>of</u> S <u>have mutually disjoint partial transversals of sizes</u> $t_1, t_2, \cdots, t_p$ <u>if and only if the inequalities</u>

$$(2.8) \qquad |S'| \geq \sum_{j=r(\overline{S}')+1}^{\infty} t_j^*$$

<u>hold for all subsets of elements</u> $S' \subseteq S$.

We shall use (2.8) in Secs. 4 and 5.

3. TRANSVERSAL COVERS

In this section we present a similar approach to the transversal covering problem described in Sec. 1. Thus we seek conditions under which the subsets $S_1, S_2, \cdots, S_m$ of S have partial transversals $T_1, T_2, \cdots, T_p$ of sizes $t_1, t_2, \cdots, t_p$ whose union $\bigcup_{i=1}^{p} T_i$ is equal to S.

We let ρ denote the maximum possible transversal size, that is, ρ is the term rank of the incidence matrix of $S_1, S_2, \cdots, S_m$ vs. $e_1, e_2, \cdots, e_n$. We shall assume throughout this section that

$$(3.1) \qquad 0 < t_1 \leq t_2 \leq \cdots \leq t_p \leq \rho.$$

Now consider the network of Fig. 2.1 with all arc orientations reversed, having v as source and u as sink. An integral flow f from v to u in this network which satisfies

$$(3.2) \qquad f(x_j, u) = c(x_j,u) = 1, \ j = 1, 2, \cdots, n,$$

produces a covering of S by partial transversals $T'_1, T'_2, \cdots, T'_p$ of sizes $t'_1 \leq t_1, t'_2 \leq t_2, \cdots, t'_p \leq t_p$. Since $t_k \leq \rho$, $k = 1, 2, \cdots, p$, each such partial transversal T'_k can be extended to a partial transversal T_k of size t_k. (This assertion is a consequence, for example, of the labeling method for constructing a maximum transversal [2; Sec. II-5]. It can also be deduced in other ways. The fact that a partial transversal can always be extended to a maximum transversal is an important one which we shall return to in Sec. 5.) Conversely, a covering by partial transversals of sizes $t_1, t_2, \cdots, t_p$ yields an integral flow f from v to u in the reversed network which satisfies arc capacities except possibly on sink arcs, where we have

$$(3.3) \qquad f(x_j,u) \geq c(x_j,u) = 1, \ j = 1, 2, \cdots, n.$$

It follows that the desired covering exists if and only if each cut separating v from u has capacity at least n.

As before, let X, Y, Z be subsets of $\{x_1, x_2, \cdots, x_n\}$, $\{y_1, y_2, \cdots, y_m\}$, and $\{z_1, z_2, \cdots, z_p\}$, respectively, and denote their complements in these sets by $\bar{X}, \bar{Y}, \bar{Z}$. The statement that all cut capacities in the reversed network are at least n becomes

$$(3.4) \qquad \sum_{z_k \in \overline{Z}} c(v,z_k) + \sum_{\substack{z_k \in Z \\ y_i \in \overline{Y}}} c(z_k,y_i) + \sum_{\substack{y_i \in Y \\ x_j \in \overline{X}}} c(y_i,x_j) + \sum_{x_j \in X} c(x_j,u) \geq n$$

for all X, Y, Z. Again we may assume that the set of arcs $(Y, \overline{X})$ is empty, so that (3.4) reduces to

$$(3.5) \qquad \sum_{z_k \in \overline{Z}} t_k + |Z| \cdot |\overline{Y}| + |X| \geq n.$$

In (3.5) we have $B(\overline{X}) \subseteq \overline{Y}$, and hence we may restrict attention to the case $\overline{Y} = B(\overline{X})$. We may also assume, for $\overline{Z}$ of fixed cardinality $|\overline{Z}| = e$, that $\overline{Z} = \{z_1, z_2, \cdots, z_e\}$. Thus (3.5) becomes

$$(3.6) \qquad \sum_{k=1}^{e} t_k + (p-e) \cdot |B(\overline{X})| \geq |\overline{X}|, \; e = 0, 1, \cdots, p,$$

$$\text{all } \overline{X} \subseteq \{x_1, x_2, \cdots, x_n\}.$$

In set-theoretic terms, (3.6) states

$$(3.7) \; \sum_{k=1}^{e} t_k + (p-e) r(S') \geq |S'|, \; e = 0,1,\cdots,p, \text{ all } S' \subseteq S.$$

The inequalities (3.7) can be further simplified to ones involving the conjugate partition $[t_j^*]$ by fixing S' and selecting e to minimize the left-hand side of

(3.7). Indeed,

$$\min_{0 \leq e \leq p} \left[\sum_{k=1}^{e} t_k + (p-e) r(S') \right] = \sum_{j=1}^{r(S')} t_j^*. \tag{3.8}$$

This establishes the following theorem.

Theorem 3.1. The subsets $S_1, S_2, \cdots, S_m$ of S have partial transversals $T_1, T_2, \cdots, T_p$ of sizes $t_1, t_2, \cdots, t_p$ such that $\bigcup_{k=1}^{p} T_k = S$ if and only if (i) $t_k \leq \rho$, $k = 1, 2, \cdots, p$, and (ii) for every $S' \subseteq S$, the inequality

$$|S'| \leq \sum_{j=1}^{r(S')} t_j^* \tag{3.9}$$

holds.

4. MAXIMUM PACKINGS AND MINIMUM COVERS

If we specialize the transversal packing and covering problems to the situation

$$t_k = t, \quad k = 1, 2, \cdots, p, \tag{4.1}$$

it is sensible to ask for maximum packings (a packing of p partial transversals of size t with p a maximum) and minimum covers (a cover by p partial transversals of size t with p a minimum).

The following corollaries are immediate consequences of Theorems 2.2 and 3.1, respectively.

Corollary 4.1. The maximum number p_{max} of partial transversals of size t in a packing is given by

$$p_{max} = \min_{S' \subseteq S,\ r(\bar{S}') < t} \left[\frac{|S'|}{t - r(\bar{S}')} \right] . \tag{4.2}$$

Corollary 4.2. The minimum number p_{min} of partial transversals of size t, $0 < t \leq \rho$, in a cover is given by

$$p_{min} = \max_{\phi \subset S' \subseteq S} \left\langle \frac{|S'|}{\min(t, r(S'))} \right\rangle . \tag{4.3}$$

In (4.2), $[x]$ denotes the largest integer $\leq x$; in (4.3), $\langle x \rangle$ denotes the least integer $\geq x$, and ϕ denotes the empty set.

5. CONNECTIONS WITH MATROID THEORY

The corollaries of Sec. 4 are closely related to two theorems of Edmonds [1: Th. 1 and Th. 2] concerning maximum packings and minimum covers composed of bases in matroids. Indeed, slightly different versions of these corollaries are special cases of Edmonds' theorems.

To make this relationship explicit, we begin by reviewing some of the notions involved. The first and most fundamental is that of "matroid," introduced by Whitney [9] and since studied intensively by Tutte [4, 5, 6, 7]

and others. For our purposes, the following characterization is convenient. A matroid is a system consisting of a finite set M of n elements $e_1, e_2, \cdots, e_n$ (frequently called the edges of M), together with subsets $I_1, I_2, \cdots, I_k$ (the independent sets of M), which satisfy the following two axioms.

Axiom 1. Every subset of an independent set is an independent set.

Axiom 2. For any subset $A \subseteq M$, all maximal independent sets contained in A have the same number of elements.

Examples of matroids abound. Two of the best-known are:

Example 1. Let M be the set of edges of a finite undirected graph, and call a set of edges independent if it contains no circuit.

Example 2. Let M be the set of columns of a matrix with elements in a field, and define independence of columns in the usual way.

Example 1 is a special case of Example 2, as may be seen by considering the vertex-edge incidence matrix of the graph, thought of as having elements in the field of integers mod 2.

Matroids may be viewed as a generalization of matrices, in which the aim is to abstract properties of linear dependence among the columns, that is, to study properties that are invariant under column permutations and elementary

row operations. but are not invariant under elementary column operations.

A maximal independent set of a matroid M is called a base of M, and the number of elements in a base is called the rank of M. We shall denote this by rank (M). Given any matroid M, the subsystem of M consisting of a subset $A \subseteq M$ and all the independent sets of M contained in A is a matroid, called the submatroid A of M. The rank, rank (A), of a set A of elements of M, is the rank of the submatroid A of M.

These definitions, and the usual notions of packing and covering, are enough to state the following two theorems of Edmonds [1]. These theorems appear to be new even for the case of matrices.

Theorem 5.1 (Edmonds). *The maximum number p_{max} of bases of a matroid M in a packing is given by*

$$p_{max} = \min_{A \subseteq M} \left[\frac{|A|}{\operatorname{rank}(M) - \operatorname{rank}(\bar{A})} \right], \tag{5.1}$$

the minimum being taken over all $A \subseteq M$ such that $\operatorname{rank}(M) - \operatorname{rank}(\bar{A}) > 0$.

Theorem 5.2 (Edmonds). *The minimum number p_{min} of bases of a matroid M in a cover is given by*

$$p_{min} = \max_{A \subseteq M} \left\langle \frac{|A|}{\operatorname{rank}(A)} \right\rangle, \tag{5.2}$$

<u>the maximum being taken over all nonempty</u> $A \subseteq M$.

Theorem 5.1 includes as special case a theorem of Tutte [8] concerning maximum packings of trees in a connected graph.

To see the precise relationship between Theorem 5.1 and Corollary 4.1, and between Theorem 5.2 and Corollary 4.2, let $S_1, S_2, \cdots, S_m$ be arbitrary subsets of set $S = \{e_1, e_2, \cdots, e_n\}$, and call a subset $T \subseteq S$ independent if T is a partial transversal of $S_1, S_2, \cdots, S_m$ of size $t' \leq t$, t being a positive integer. It follows from the fact, mentioned in Sec. 3, that a partial transversal of size t' can be extended to one of size t'', $t' < t'' \leq \rho$, that this definition of independence makes S into a matroid $S(t)$.* It is the matroid $S(t)$ which is relevant in making the connection between Theorem 5.1 and Corollary 4.1, and between Theorem 5.2 and Corollary 4.2.

We first consider Theorem 5.2 and Corollary 4.2. In matroid $S(t)$ the rank of $S' \subseteq S$ is given by

$$(5.3) \qquad \text{rank}\,(S') = \min\,(t, \rho(S')) \leq \min\,(t, r(S')),$$

where $\rho(S')$ is the term rank of the incidence matrix of $S_1, S_2, \cdots, S_m$ vs. elements of S'. Thus the denominators

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*The matroid $S(\rho)$ is but one of a considerably larger class of matroids recently found by Edmonds. The fact that $S(\rho)$ is a matroid implies that $S(t)$ is also, since it is not difficult to show that, given any matroid M, other matroids are obtained by taking as independent sets those independent sets of M which have at most t elements, $t = 1, 2, \cdots, \text{rank}\,(M) - 1$.

in (5.2) and (4.3) can well be different. However, it can be shown, using the König theorem on term rank, that if inequality holds in (5.3), there is a subset $S'' \subseteq S'$ satisfying

$$(5.4) \qquad \left\langle \frac{|S''|}{\min(t, r(S''))} \right\rangle \geq \left\langle \frac{|S'|}{\mathrm{rank}(S')} \right\rangle .$$

Thus Corollary 4.2 is an instance of Theorem 5.2.

Similarly Corollary 4.1 is a special case of Theorem 5.1.

In view of these connections, it is reasonable to ask if analogous generalizations of Theorems 2.2 and 3.1 hold for matroids. It seems likely that the conditions

$$(5.5) \qquad |A| \geq \sum_{j=\mathrm{rank}(\bar{A})+1}^{\infty} t_j^*, \quad \text{all } A \subseteq M,$$

are necessary and sufficient for the existence of mutually disjoint independent sets of sizes $t_1, t_2, \cdots, t_p$ in matroid M, and that the conditions

$$(5.6) \qquad |A| \leq \sum_{j=1}^{\mathrm{rank}(A)} t_j^*, \quad \text{all } A \subseteq M,$$

are necessary and sufficient for a covering of the edges of matroid M by independent sets of sizes $t_1, t_2, \cdots, t_p$, where each $t_i \leq \mathrm{rank}\ (M)$. We note that the necessity of these conditions is easily established. First consider

(5.6). Let $I_1, I_2, \cdots, I_p$ be a covering by independent sets of sizes $t_1, t_2, \cdots, t_p$. Then, for $A \subseteq M$,

$$|A| \leq \sum_{i=1}^{p} |A \cap I_i| \leq \sum_{i=1}^{p} \min\,(\text{rank}\,(A),\, t_i) = \sum_{j=1}^{\text{rank}(A)} t_j^*.$$

Similarly let $I_1, I_2, \cdots, I_p$ be a packing of independent sets of sizes $t_1, t_2, \cdots, t_p$, and let A be a subset of M. Then

$$\text{rank}\,(A \cap I_i) + \text{rank}\,(\bar{A} \cap I_i) \geq \text{rank}\,(I_i) = t_i,$$

and hence

$$|A \cap I_i| + \text{rank}\,(\bar{A} \cap I_i) \geq t_i.$$

Thus

$$|A| \geq \sum_{i=1}^{p} |A \cap I_i| \geq \sum_{i=1}^{p} (t_i - \text{rank}\,(\bar{A} \cap I_i)),$$

from which it follows that

$$|A| \geq \sum_{i=1}^{p} (t_i - \min\,(\text{rank}\,(\bar{A}),\, t_i)) = \sum_{j=\text{rank}(\bar{A})+1}^{\infty} t_j^*.$$

Note also that sufficiency of the conditions (5.5) and (5.6) would imply Edmonds' Theorems 5.1 and 5.2, respectively.

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